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ON THE STUDY OF BIFURCATING RIGHT FIBONOMIAL NUMBERS AND RB-TRINOMIAL NUMBERS

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ABSTRACT
Fontené once introduced a generalized form of binomial coefficients by substituting natural numbers with terms from an arbitrary sequence $\{A_n\}$ of real or complex numbers, which he referred to as Fibonomial coefficients. Since then, significant interest has developed around Fibonomial numbers which is two dimensional in which n is divided into two parts, particularly when the sequence $\{A_n\}$ is chosen as $\{F_n\}$, the well-known Fibonacci sequence. More recently, researchers have explored a further extension by considering $\{A_n\} = \{F_n^R\}$, the sequence of right Fibonacci numbers. In this paper, we take this generalization a step further by defining Fibonomial coefficients based on the sequence $\{A_n\} = \{F_n^{R(a,b)}\}$, known as the right Bifurcating Fibonacci numbers. Also, there were a new generalization was established for three-dimensional Fibonomial numbers which is the extension of n divided into three parts, known as F-trinomial numbers. In this paper, we choose right bifurcating Fibonacci sequence and introduced RB-trinomial numbers. Then, we derive several identities associated with both of them. Additionally, we examine some of their bounds for both numbers.

KEYWORDS: *Binomial Coefficients, Fibonacci Numbers, Bifurcating Fibonacci numbers, Fibonomial Coefficients, Trinomial Coefficients, Characteristic equation.*
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1.INTRODUCTION

In 2003, Kalman and Mena [8] conducted a study on the sequence of generalized Fibonacci numbers, which are defined as follows:

Definition: For any positive integers a and b, the *generalized Fibonacci numbers* $\left\{F_n^{(a,b)}\right\}$ are defined by the recurrence relation $F_n^{(a,b)} = aF_{n-1}^{(a,b)} + bF_{n-2}^{(a,b)};\ n \geq 2$, where $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$.

First few terms of the sequence $\left\{F_n^{(a,b)}\right\}$ are $0, 1, a, a^2 + b, a^3 + 2ab, a^4 + 3a^2b + b^2, \ldots$. Clearly, we can easily see that $F_n^{(1,1)} = F_n$, the classical n[th] Fibonacci number.

Additionally, Shah and Diwan [3] investigated a generalized form of the Fibonacci sequence, referred to as the *right generalized Fibonacci sequence*. This sequence, constructed through bifurcation, follows a recurrence relation that incorporates two real parameters, a and b, and is defined as follows.

$$F_n^{R(a,b)} = \begin{cases} F_{n-1}^{R(a,b)} + aF_{n-2}^{R(a,b)}\ ; \text{when } n \text{ is odd} \\ F_{n-1}^{R(a,b)} + bF_{n-2}^{R(a,b)}\ ; \text{when } n \text{ is even} \end{cases}, \text{ where } F_0^{R(a,b)} = 0, F_1^{R(a,b)} = 1 \qquad (1)$$

First few terms of this sequence are 0, 1, 1, 1+a, 1+a+b, $1 + 2a + b + a^2, \ldots$. Also, the extended Binet formula for this sequence is $F_n^{R(a,b)} = \frac{\gamma^{\chi(n)}\alpha^{\left[\frac{n}{2}\right]} - \delta^{\chi(n)}\beta^{\left[\frac{n}{2}\right]}}{\alpha-\beta}$; where, $\alpha - b = \gamma, \beta - b = \delta, \alpha = \frac{(a+b+1)+\sqrt{(a+b+1)^2+4ab}}{2}, \beta = \frac{(a+b+1)-\sqrt{(a+b+1)^2+4ab}}{2}$ and $\chi(n) = \begin{cases} 0; if\ n\ is\ even. \\ 1; if\ n\ is\ odd. \end{cases}$ Then we can clearly see that $F_n^{R(1,1)} = F_n$, the classical n[th] Fibonacci number.

Benjamin and Plott [1] expanded on the concept of Fibonomial numbers and explored various intriguing properties related to them in 2008. Later, in 2018, Shah and Shah [13] introduced Genomial numbers by generalizing the idea, replacing Fibonacci numbers with k-Fibonacci numbers. Building on this foundation, our study further extends this concept by incorporating Right Bifurcating Fibonomial numbers and also, in 2024, Shah and Shah [14] derived F-trinomial numbers which were extension of Fibonomial numbers as n is divided into three parts, parallelly in this paper we defined RB-trinomial numbers, both with the use of

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right bifurcating Fibonacci sequence. We next establish multiple identities and derive significant results associated with these sequences.

2.RIGHT BIFURCATING FIBONOMIAL NUMBERS

Fontené [6] published a work on Binomial coefficients, in 1915, where he replaced integers within the coefficients with elements from a variable sequence denoted as $\{A_n\}$. This innovation sparked growing interest in exploring Fibonomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$, where the sequence $\{A_n\}$ was substituted by the Fibonacci sequence $\{F_n\}$. Specifically, $\begin{bmatrix} n \\ k \end{bmatrix}$ is defined as $\frac{F_n^*}{F_k^* \times F_{n-k}^*}$, where $F_n^* = F_n \times F_{n-1} \times F_{n-2} \times \ldots \times F_2 \times F_1$ and $F_0^* = 1$.

In 2014, Koshy [9] introduced Pellnomial numbers by replacing $\{A_n\}$ with $\{P_n\}$, the sequence of Pell numbers, and highlighted various properties associated with them. Later, Shah and Shah [12] extended this concept, defining Genomial numbers as follows, $\begin{bmatrix} n \\ r \end{bmatrix} = \frac{F_n^{(a,1)*}}{F_r^{(a,1)*} \times F_{n-r}^{(a,1)*}}$; $0 \le r \le n$, where $F_n^{(a,1)*} = F_n^{(a,1)} \times F_{n-1}^{(a,1)} \times \cdots \times F_2^{(a,1)} \times F_1^{(a,1)}$ and $F_0^{(a,1)*} = 1$.

Recently, Desai and Shah [2] introduced the proper right a-Fibonomial numbers, denoted by $\begin{bmatrix} m \\ k \end{bmatrix}_R$, *and defined as* $\begin{bmatrix} m \\ k \end{bmatrix}_R = \frac{F_m^{(1,a)*}}{F_k^{(1,a)*} \times F_{m-k}^{(1,a)*}}$, *where* $0 \le k \le m$. *Here,* $F_m^{(1,a)*} = F_m^{(1,a)} \times F_{m-1}^{(1,a)} \times F_{m-2}^{(1,a)} \times \cdots \times F_1^{(1,a)}$ and $F_0^{(1,a)*} = 1$.

In this work, we expand the investigation to include the Right Bifurcating Fibonomial numbers, defined in the following way.

Definition: The *Right Bifurcating Fibonomial numbers* $\begin{bmatrix} n \\ k \end{bmatrix}_{RB}$ are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{RB} = \frac{F_n^{R(a,b)*}}{F_k^{R(a,b)*} \times F_{n-k}^{R(a,b)*}} ; 0 \le k \le n$$

where $F_n^{R(a,b)*} = F_n^{R(a,b)} \times F_{n-1}^{R(a,b)} \times F_{n-2}^{R(a,b)} \times \cdots \times F_1^{R(a,b)}$ and $F_0^{R(a,b)*} = 1$.

Table 1. Right Bifurcating Fibonomial number triangle

						1						
					1		1					
				1		1		1				
			1		$1+a$		$1+a$		1			
		1		$1+a+b$		$1+a+b$		$1+a+b$		1		
	1		$1+2a+a^2+b$		$(1+a+b)(1+2a+a^2+b)$		$(1+a+b)(1+2a+a^2+b)$		$1+2a+a^2+b$		1	
1		$1+2a+2b+a^2+b^2+ab$		$(1+2a+2b+a^2+b^2+ab)(1+2a+a^2+b)$		$[(1+a)^2+2b+b^2+ab)((1+a)^2+b)(1+a+b)]/(1+a)$		$(1+2a+2b+a^2+b^2+ab)(1+2a+a^2+b)$		$1+2a+2b+a^2+b^2+ab$		1

This table shows the triangle of Right bifurcating Fibonomial numbers for $n = 7$, we can extend it for more values of n.

The following results are obviously obtained from the definition.

Lemma 2.1. (i) $\begin{bmatrix} n \\ 0 \end{bmatrix}_{RB} = 1 = \begin{bmatrix} n \\ n \end{bmatrix}_{RB}$.

(ii) $\begin{bmatrix} n \\ 1 \end{bmatrix}_{RB} = F_n^{R(a,b)}$.

(iii) $\begin{bmatrix} n \\ n-k \end{bmatrix}_{RB} = \begin{bmatrix} n \\ k \end{bmatrix}_{RB}$.

By the use of the definition of Right bifurcating Fibonomial numbers, we can easily prove the following identities.

Lemma 2.2. The recurrence relation for right bifurcating Fibonomial numbers is $\begin{bmatrix} n \\ k \end{bmatrix}_{RB} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{RB} \times \frac{F_n^{R(a,b)}}{F_k^{R(a,b)}}$.

Another recurrence relation for right bifurcating Fibonomial numbers is obtained as follows:

Corollary 2.1. $\begin{bmatrix} n \\ k \end{bmatrix}_{RB} = \begin{bmatrix} n \\ k-1 \end{bmatrix}_{RB} \times \frac{F_{n-k+1}^{R(a,b)}}{F_k^{R(a,b)}}$.

From lemma 2.2 and corollary 2.1, the following lemma follows immediately.

Lemma 2.3. $\begin{bmatrix} n \\ k \end{bmatrix}_{RB} F_{n-k}^{L(a,b)} = \begin{bmatrix} n-1 \\ k \end{bmatrix}_{RB} F_n^{L(a,b)}$

In [7], Gould gave an interesting result known as *Star of David theorem* for binomial coefficients, stated as $\binom{n-a}{r-a}\binom{n}{r+a}\binom{n+a}{r} = \binom{n-a}{r}\binom{n+a}{r+a}\binom{n}{r-a}$.

Theorem 2.1. The Gould's identity for right bifurcating Fibonomial numbers is $\begin{bmatrix} n-r \\ k-r \end{bmatrix}_{RB} \begin{bmatrix} n \\ k+r \end{bmatrix}_{RB} \begin{bmatrix} n+r \\ k \end{bmatrix}_{RB} = \begin{bmatrix} n-r \\ k \end{bmatrix}_{RB} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{RB} \begin{bmatrix} n \\ k-r \end{bmatrix}_{RB}$.

Proof: Using definition of right bifurcating Fibonomial numbers, we get, $\begin{bmatrix} n-r \\ k-r \end{bmatrix}_{RB} \begin{bmatrix} n \\ k+r \end{bmatrix}_{RB} \begin{bmatrix} n+r \\ k \end{bmatrix}_{RB} = \frac{F_{n-r}^{R(a,b)*}}{F_{k-r}^{R(a,b)*} \times F_{n-k}^{R(a,b)*}} \times \frac{F_n^{R(a,b)*}}{F_{k+r}^{R(a,b)*} \times F_{n-k-r}^{R(a,b)*}} \times \frac{F_{n+r}^{R(a,b)*}}{F_k^{R(a,b)*} \times F_{n+r-k}^{R(a,b)*}}$

$$= \frac{F_{n-r}^{R(a,b)*}}{F_k^{R(a,b)*} \times F_{n-k-r}^{R(a,b)*}} \times \frac{F_{n+r}^{R(a,b)*}}{F_{k+r}^{R(a,b)*} \times F_{n-k}^{R(a,b)*}} \times \frac{F_n^{R(a,b)*}}{F_{k-r}^{R(a,b)*} \times F_{n+r-k}^{R(a,b)*}}$$

$$= \begin{bmatrix} n-r \\ k \end{bmatrix}_{RB} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{RB} \begin{bmatrix} n \\ k-r \end{bmatrix}_{RB}.$$ □

Corollary 2.2. Special generalization of Gould's identity with $r = 1$ is $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{RB} \begin{bmatrix} n \\ k+1 \end{bmatrix}_{RB} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{RB} = \begin{bmatrix} n-1 \\ k \end{bmatrix}_{RB} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{RB} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{RB}$.

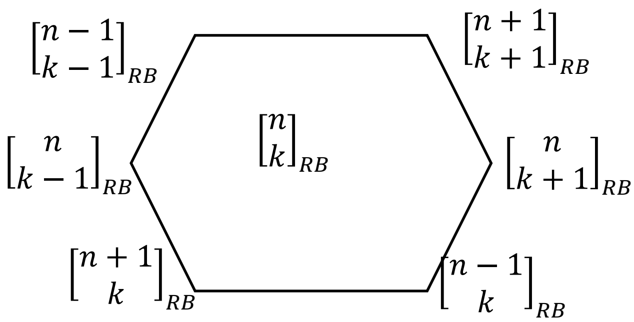

Figure 1. Star of David for Right bifurcating Fibonomial numbers

This figure illustrates the Right Bifurcating Fibonomial number $\begin{bmatrix} n \\ k \end{bmatrix}_{RB}$, surrounded by all the neighboring Right Bifurcating Fibonomial numbers within a hexagonal arrangement.

2.1 Bounds for Right bifurcating Fibonomial numbers

Given the root of right bifurcating Fibonacci numbers, defined as $\alpha = \frac{(a+b+1)+\sqrt{(a+b+1)^2-4ab}}{2}$ which satisfies the characteristic equation $abx^2 - (a+b+1)x + 1 = 0$ of $F_n^{R(a,b)}$, we utilize this root to establish bounds for right bifurcating Fibonacci numbers.

Lemma 2.4. For right bifurcating Fibonacci numbers, the bounds is $\alpha^{n-5}\left(a^{\chi(n)}b^{1-\chi(n)} + \alpha\right) \le F_n^{R(a,b)} \le \alpha^{n-4}\left(a^{\chi(n)}b^{1-\chi(n)} + \alpha\right)$; $\chi(n) = \begin{cases} 0; if\ n\ is\ even \\ 1; if\ n\ is\ odd \end{cases}$.

Proof: We establish the result by applying the principle of mathematical induction on n.

Consider, $P(n)$: $\alpha^{n-5}\left(a^{\chi(n)}b^{1-\chi(n)} + \alpha\right) \le F_n^{R(a,b)} \le \alpha^{n-4}\left(a^{\chi(n)}b^{1-\chi(n)} + \alpha\right)$. Now, for $n = 4$, we have $F_4^{R(a,b)} = a + b + 1$. Also $\alpha^2 = \frac{\left((a+b+1)+\sqrt{a^2b^2+4ab}\right)^2}{4} = (a+b+1)\alpha - ab$.

Therefore, it is clear that: $F_4^{R(a,b)} \le \alpha^2$. Also, $\frac{a+b+1}{2} \le a + b + 1$ is always true. This is same as $\frac{(a+b+1)+\sqrt{(a+b+1)^2-4ab}}{2} \le a + b + 1$. Therefore, $\alpha \le F_4^{R(a,b)}$. Thus, we have $\alpha \le F_4^{R(a,b)} \le \alpha^2$. That is, $P(n)$ is true for $n = 4$.

Next here we assume that $P(n)$ is true for all integers k, such that $4 \le k \le n$. Thus, we have $\alpha^{k-3} \le F_k^{R(a,b)} \le \alpha^{k-2}$ and $\alpha^{k-4} \le F_{k-1}^{R(a,b)} \le \alpha^{k-3}$. Now, consider k is even integer as $k = 2x; x \in \mathbb{Z}$ from these we can have two identities as $\alpha^{2x-3} \le F_{2x}^{R(a,b)} \le \alpha^{2x-2}$ and $\alpha^{2x-4} \le F_{2x-1}^{R(a,b)} \le \alpha^{2x-3}$. Now by combining both identities, we have $\alpha^{2x-3} + b\alpha^{2x-4} \le F_{2x}^{R(a,b)} + bF_{2x-1}^{R(a,b)} \le \alpha^{2x-2} + b\alpha^{2x-3}$. Using the definition of right bifurcating Fibonacci numbers, we now get $\alpha^{k-4}(\alpha + b) \le F_{k+1}^{R(a,b)} \le \alpha^{k-3}(\alpha + b)$.

Similarly, if k is odd integer as $k = 2x + 1; x \in \mathbb{Z}$ then we can have two identities related to α and $F_n^{R(a,b)}$ as $\alpha^{2x-2} \le F_{2x+1}^{R(a,b)} \le \alpha^{2x-1}$ and $\alpha^{2x-3} \le F_{2x}^{R(a,b)} \le \alpha^{2x-2}$. By merging these identities once again, we obtain $\alpha^{2x-2} + a\alpha^{2p-3} \le F_{2x+1}^{R(a,b)} + F_{2x}^{R(a,b)} \le \alpha^{2x-1} + a\alpha^{2x-2}$. Also, using the definition of right bifurcating Fibonacci numbers, we get $\alpha^{k-4}(\alpha + a) \le F_{k+1}^{R(a,b)} \le \alpha^{k-3}(\alpha + a)$. That is, $P(n)$ holds for $k + 1$ also. Thus, by the principle of mathematical induction, $P(n)$ holds for all $n \ge 4$.

This result enables us to determine the bounds for right bifurcating Fibonomial numbers.

Theorem 2.2. For right bifurcating Fibonomial numbers, the bounds are $\alpha^{k(n-k-1)} \le \begin{bmatrix} n \\ k \end{bmatrix}_{RB} \le \alpha^{k(n-k+1)}; n \in Z$.

Proof: It is widely recognized that $\alpha^{n-5}(\alpha + b) \le F_n^{R(a,b)} \le \alpha^{n-4}(\alpha + b)$; $n =$ even. Therefore, $\alpha^{m-t-5}(\alpha + b) \le F_{m-t}^{R(a,b)} \le \alpha^{m-t-4}(\alpha + b)$ and $\alpha^{t-4}(\alpha + b) \le F_{t+1}^{R(a,b)} \le \alpha^{t-3}(\alpha + b)$. Thus, we get $\alpha^{m-2t-2} \le \frac{F_{m-t}^{R(a,b)}}{F_{t+1}^{R(a,b)}} \le \alpha^{m-2t}$.

Now, based on the definition of right bifurcating Fibonomial numbers, we obtain

$$\begin{bmatrix} n \\ k \end{bmatrix}_{RB} = \frac{F_n^{R(a,b)} \times \cdots \times F_{n-k+1}^{R(a,b)}}{F_1^{R(a,b)} \times F_2^{R(a,b)} \times \cdots \times F_k^{R(a,b)}} \le \alpha^{n+(n-2)+\cdots+(n-2k+2)} = \alpha^{k(n-k+1)}.$$

Also,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{RB} = \frac{F_n^{R(a,b)} \times \cdots \times F_{n-k+1}^{R(a,b)}}{F_1^{R(a,b)} \times F_2^{R(a,b)} \times \cdots \times F_k^{R(a,b)}} \ge \alpha^{(n-2)+(n-4)+\cdots+(n-2k-2)} = \alpha^{k(n-k-1)}.$$

Thus, $\alpha^{k(n-k-1)} \le \begin{bmatrix} n \\ k \end{bmatrix}_{RB} \le \alpha^{k(n-k+1)}; n =$ even.

In a similar manner, we can derive $\alpha^{k(n-k-1)} \le \begin{bmatrix} n \\ k \end{bmatrix}_{RB} \le \alpha^{k(n-k+1)}$ for $n =$ odd; as required.

□

3. RB-TRINOMIAL NUMBERS

In 2024, Shah and Shah [14] introduced F-trinomial numbers for three-dimensional extension as they divided n into three parts. For any positive integer n and non-negative integers r, s, t such that $r + s + t = n$ and $F_n^* = F_n \times F_{n-1} \times \cdots \times F_1$, the *F-trinomial numbers* are defined as

$$\begin{bmatrix} n \\ r, s, t \end{bmatrix}_F = \frac{F_n^*}{F_r^* F_s^* F_t^*}$$

Furthermore, we refer to the resulting values as *RB-trinomial numbers*, derived in the context of right bifurcating Fibonacci sequences.

Definition: For any positive integer n and non-negative integers u, v, w such that $u + v + w = n$ and $F_n^{R(a,b)^*} = F_n^{R(a,b)} \times F_{n-1}^{R(a,b)} \times \cdots \times F_1^{R(a,b)}$, the *RB-trinomial numbers* are defined as

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$$\begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB} = \frac{F_n^{R(a,b)^*}}{F_u^{R(a,b)^*} F_v^{R(a,b)^*} F_w^{R(a,b)^*}}$$

The following results are trivial from the definition of RB-trinomial numbers.

Lemma 3.1. $\begin{bmatrix} n \\ 0,v,w \end{bmatrix}_{RB} = \begin{bmatrix} n \\ v \end{bmatrix}_{RB}$; regular right bifurcating Fibonomial number.

Lemma 3.2. $\begin{bmatrix} n \\ u,0,0 \end{bmatrix}_{RB} = 1$

Lemma 3.3. $\begin{bmatrix} n \\ 1,v,w \end{bmatrix}_{RB} = F_n^{R(a,b)} \begin{bmatrix} n-1 \\ v \end{bmatrix}_{RB}$

In this instance, the RB-trinomial number emerges as the result of multiplying a right bifurcating Fibonacci number by its corresponding right bifurcating Fibonomial coefficient.

Theorem 3.1. RB-trinomial numbers are always integers.

Proof: From the definition of RB-trinomial numbers, we have

$$\begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB} = \frac{F_n^{R(a,b)^*}}{F_u^{R(a,b)^*} F_v^{R(a,b)^*} F_w^{R(a,b)^*}} = \frac{\overbrace{F_n^{R(a,b)} \times F_{n-1}^{R(a,b)} \times \cdots \times F_{v+w+1}^{R(a,b)}} \times \overbrace{F_{v+w}^{R(a,b)} \times \cdots \times F_{w+1}^{R(a,b)}} \times \overbrace{F_w^{R(a,b)} \times \cdots \times F_1^{R(a,b)}}}{F_u^{R(a,b)^*} F_v^{R(a,b)^*} F_w^{R(a,b)^*}}$$

This fraction contains u, v and w number of consecutive right bifurcating Fibonacci numbers in the numerator as well as in denominator. Since multiplication of any $'n'$ consecutive right bifurcating Fibonacci numbers is always divisible by the multiplication of first $'n'$ consecutive right bifurcating Fibonacci numbers [11], it is now evident that $\begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB}$ is always integer. ☐

Again from [7], Gould's well-known result, *Star of David theorem* for binomial coefficients, we arrived at the following identity related to the RB-trinomial numbers:

Theorem 3.2. The Gould's identity for RB-trinomial numbers is

$$\begin{bmatrix} n-1 \\ u,v-1,w \end{bmatrix}_{RB} \begin{bmatrix} n \\ u-1,v,w+1 \end{bmatrix}_{RB} \begin{bmatrix} n \\ u+1,v,w-1 \end{bmatrix}_{RB} \begin{bmatrix} n+1 \\ u,v+1,w \end{bmatrix}_{RB}$$
$$= \begin{bmatrix} n-1 \\ u-1,v,w \end{bmatrix}_{RB} \begin{bmatrix} n \\ u,v-1,w+1 \end{bmatrix}_{RB} \begin{bmatrix} n \\ u,v+1,w-1 \end{bmatrix}_{RB} \begin{bmatrix} n+1 \\ u+1,v,w \end{bmatrix}_{RB}$$
$$= \begin{bmatrix} n-1 \\ u,v,w-1 \end{bmatrix}_{RB} \begin{bmatrix} n \\ u-1,v+1,w \end{bmatrix}_{RB} \begin{bmatrix} n \\ u+1,v-1,w \end{bmatrix}_{RB} \begin{bmatrix} n+1 \\ u,v,w+1 \end{bmatrix}_{RB}$$

Proof: Using the definition of RB-trinomial numbers, we have

$$\begin{bmatrix} n-1 \\ u,v-1,w \end{bmatrix}_{RB} \begin{bmatrix} n \\ u-1,v,w+1 \end{bmatrix}_{RB} \begin{bmatrix} n \\ u+1,v,w-1 \end{bmatrix}_{RB} \begin{bmatrix} n+1 \\ u,v+1,w \end{bmatrix}_{RB}$$

$$= \frac{F_{n-1}^{R(a,b)^*}}{F_u^{R(a,b)^*} F_{v-1}^{R(a,b)^*} F_w^{R(a,b)^*}} \times \frac{F_n^{R(a,b)^*}}{F_{u-1}^{R(a,b)^*} F_v^{R(a,b)^*} F_{w+1}^{R(a,b)^*}} \times \frac{F_n^{R(a,b)^*}}{F_{u+1}^{R(a,b)^*} F_v^{R(a,b)^*} F_{w-1}^{R(a,b)^*}} \times \frac{F_{n+1}^{R(a,b)^*}}{F_u^{R(a,b)^*} F_{v+1}^{R(a,b)^*} F_w^{R(a,b)^*}}$$

$$= \frac{F_{n-1}^{R(a,b)^*}}{F_{u-1}^{R(a,b)^*} F_v^{R(a,b)^*} F_w^{R(a,b)^*}} \times \frac{F_n^{R(a,b)^*}}{F_u^{R(a,b)^*} F_{v-1}^{R(a,b)^*} F_{w+1}^{R(a,b)^*}} \times \frac{F_n^{R(a,b)^*}}{F_u^{R(a,b)^*} F_{v+1}^{R(a,b)^*} F_{w-1}^{R(a,b)^*}} \times \frac{F_{n+1}^{R(a,b)^*}}{F_{u+1}^{R(a,b)^*} F_v^{R(a,b)^*} F_w^{R(a,b)^*}}$$

$$= \begin{bmatrix} n-1 \\ u-1,v,w \end{bmatrix}_{RB} \begin{bmatrix} n \\ u,v-1,w+1 \end{bmatrix}_{RB} \begin{bmatrix} n \\ u,v+1,w-1 \end{bmatrix}_{RB} \begin{bmatrix} n+1 \\ u+1,v,w \end{bmatrix}_{RB}.$$

Again, using the definition of RB-trinomial numbers, we have

$$\begin{bmatrix} n-1 \\ u,v-1,w \end{bmatrix}_{RB} \begin{bmatrix} n \\ u-1,v,w+1 \end{bmatrix}_{RB} \begin{bmatrix} n \\ u+1,v,w-1 \end{bmatrix}_{RB} \begin{bmatrix} n+1 \\ u,v+1,w \end{bmatrix}_{RB}$$

$$= \frac{F_{n-1}^{R(a,b)^*}}{F_u^{R(a,b)^*} F_{v-1}^{R(a,b)^*} F_w^{R(a,b)^*}} \times \frac{F_n^{R(a,b)^*}}{F_{u-1}^{R(a,b)^*} F_v^{R(a,b)^*} F_{w+1}^{R(a,b)^*}} \times \frac{F_n^{R(a,b)^*}}{F_{u+1}^{R(a,b)^*} F_v^{R(a,b)^*} F_{w-1}^{R(a,b)^*}} \times \frac{F_{n+1}^{R(a,b)^*}}{F_u^{R(a,b)^*} F_{v+1}^{R(a,b)^*} F_w^{R(a,b)^*}}$$

$$= \frac{F_{n-1}^{R(a,b)^*}}{F_u^{R(a,b)^*}F_v^{R(a,b)^*}F_{w-1}^{R(a,b)^*}} \times \frac{F_n^{R(a,b)^*}}{F_{u-1}^{R(a,b)^*}F_{v+1}^{R(a,b)^*}F_w^{R(a,b)^*}} \times \frac{F_n^{R(a,b)^*}}{F_{u+1}^{R(a,b)^*}F_{v-1}^{R(a,b)^*}F_w^{R(a,b)^*}} \times \frac{F_{n+1}^{R(a,b)^*}}{F_u^{R(a,b)^*}F_v^{R(a,b)^*}F_{w+1}^{R(a,b)^*}}$$

$$= \begin{bmatrix} n-1 \\ u,v,w-1 \end{bmatrix}_{RB} \begin{bmatrix} n \\ u-1,v+1,w \end{bmatrix}_{RB} \begin{bmatrix} n \\ u+1,v-1,w \end{bmatrix}_{RB} \begin{bmatrix} n+1 \\ u,v,w+1 \end{bmatrix}_{RB},$$ as desired. □

3.1 Bounds for RB-trinomial numbers

We consider the bounds of RB-trinomial numbers in terms of $\alpha = \frac{(a+b+1)+\sqrt{(a+b+1)^2-4ab}}{2}$ which is the root of the root of characteristic equation $abx^2 - (a+b+1)x + 1 = 0$ of $F_n^{R(a,b)}$.

Theorem 3.3. The bounds for RB-trinomial numbers is

$$[a^{\chi(n)}b^{1-\chi(n)} + \alpha]^{\frac{w}{2}}\alpha^{(w-4)(u+v)+uv} \le \begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB} \le [a^{\chi(n)}b^{1-\chi(n)} + \alpha]^{\frac{w}{2}}\alpha^{(w-3)(u+v)+uv};$$

$n =$ even with $v = w =$ even and $n =$ odd with u or $v =$ odd. And

$$(a+\alpha)^{\frac{w-2}{2}}(b+\alpha)^{\frac{w+2}{2}}\alpha^{(w-4)(u+v)+uv} \le \begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB} \le (a+\alpha)^{\frac{w-2}{2}}(b+\alpha)^{\frac{w+2}{2}}\alpha^{(w-3)(u+v)+uv};$$

$n =$ even with $u = v =$ odd.

Proof: Since $\alpha^{n-5}\left(a^{\chi(n)}b^{1-\chi(n)} + \alpha\right) \le F_n^{R(a,b)} \le \alpha^{n-4}\left(a^{\chi(n)}b^{1-\chi(n)} + \alpha\right)$;

$\chi(n) = \begin{cases} 0; if\ n\ is\ even \\ 1; if\ n\ is\ odd \end{cases}$; for all $n \ge 4$, from the definition of RB-trinomial numbers, we have

$$\begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB} = \frac{F_n^{R(a,b)} \times F_{n-1}^{R(a,b)} \times \cdots \times F_1^{R(a,b)}}{\left(F_u^{R(a,b)} \times F_{u-1}^{R(a,b)} \times \cdots \times F_1^{R(a,b)}\right)\left(F_v^{R(a,b)} \times F_{v-1}^{R(a,b)} \times \cdots \times F_1^{R(a,b)}\right)\left(F_w^{R(a,b)} \times F_{w-1}^{R(a,b)} \times \cdots \times F_1^{R(a,b)}\right)}$$
$$= \frac{F_n^{R(a,b)} \times F_{n-1}^{R(a,b)} \times \cdots \times F_{n-u-v+1}^{R(a,b)}}{\left(F_u^{R(a,b)} \times F_{u-1}^{R(a,b)} \times \cdots \times F_1^{R(a,b)}\right)\left(F_v^{R(a,b)} \times F_{v-1}^{R(a,b)} \times \cdots \times F_1^{R(a,b)}\right)}.$$

Case: 1 $n =$ even and $u = v =$ even.

$$[(a+\alpha)(b+\alpha)]^{\frac{n-u-v}{2}} \frac{\alpha^{(n-5)+(n-6)+\cdots+(n-u-v-4)}}{\left(\alpha^{(u-5)+(u-6)+\cdots+0}\right)\left(\alpha^{(v-5)+(v-6)+\cdots+0}\right)} \le \begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB}$$
$$\le [(a+\alpha)(b+\alpha)]^{\frac{n-u-v}{2}} \frac{\alpha^{(n-4)+(n-5)+\cdots+(n-u-v-3)}}{\left(\alpha^{(u-4)+(u-5)+\cdots+(-1)}\right)\left(\alpha^{(v-4)+(v-5)+\cdots+(-1)}\right)}$$

Thus,

$$[a^{\chi(n)}b^{1-\chi(n)} + \alpha]^{\frac{w}{2}}\alpha^{(w-4)(u+v)+uv} \le \begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB} \le [a^{\chi(n)}b^{1-\chi(n)} + \alpha]^{\frac{w}{2}}\alpha^{(w-3)(u+v)+uv}$$

Case: 2 $n =$ even and $u = v =$ odd.

$$(a+\alpha)^{\frac{n-v-u-2}{2}}(b+\alpha)^{\frac{n-v-u+2}{2}}\alpha^{(w-4)(u+v)+uv} \le \begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB} \le (a+\alpha)^{\frac{n-v-u-2}{2}}(b+\alpha)^{\frac{n-v-u+2}{2}}\alpha^{(w-3)(u+v)+uv}$$

Thus,

$$(a+\alpha)^{\frac{w-2}{2}}(b+\alpha)^{\frac{w+2}{2}}\alpha^{(w-4)(u+v)+uv} \le \begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB} \le (a+\alpha)^{\frac{w-2}{2}}(b+\alpha)^{\frac{w+2}{2}}\alpha^{(w-3)(u+v)+uv}$$

Case: 3 $n =$ odd and u or $v =$ odd.

$$[(a+\alpha)(b+\alpha)]^{\frac{n-v-u}{2}}\alpha^{(w-4)(u+v)+uv} \le \begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB} \le [(a+\alpha)(b+\alpha)]^{\frac{n-v-u}{2}}\alpha^{(w-3)(u+v)+uv}$$

Thus,

$$[a^{\chi(n)}b^{1-\chi(n)} + \alpha]^{\frac{w}{2}}\alpha^{(w-4)(u+v)+uv} \le \begin{bmatrix} n \\ u,v,w \end{bmatrix}_{RB} \le [a^{\chi(n)}b^{1-\chi(n)} + \alpha]^{\frac{w}{2}}\alpha^{(w-3)(u+v)+uv},$$

as required. □

4. CONCLUSION

This paper introduces right bifurcating Fibonomial numbers, RB-trinomial numbers and presents several related results obtained from their analysis.

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